MATH 127 - Midterm Exam 2 - Review

Spring 2022 - Sections 14.8, 15.1-15.6, and 13.1-13.3 and some 16.1

Midterm Exam 2, Tuesday 4/12, 5:50-7:50 pm in Budig 130

The following is a list of important concepts will be tested on Midterm Exam 2. This is not a complete list of the material that you should know for the course, but the review provides a summary of concepts and the problems are a good indication of what will be emphasized on the free response portion of the exam. A thorough understanding of all of the following concepts will help you perform well on the exam. Some places to find problems on these topics are the following: in the book, in the slides, in the homework, on quizzes, and Achieve.

Vector Valued Functions: (Sections 13.1 - 13.3)

A **vector function** has scalar inputs and vector outputs.

In $\mathbb{R}^2$: $\vec{s}(t) = \langle f(t), g(t) \rangle$ In $\mathbb{R}^3$: $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ f, g, h are scalar functions.

The range of $\vec{r}(t)$ in $\mathbb{R}^3$ is a space curve $\mathcal{C}$. Every space curve can be parameterized in infinitely many ways. An arclength parametrization of $\mathcal{C}$ has speed 1 everywhere, $\mathbf{s}(t) = \left|\vec{r}\,'(t)\right| = 1$ for all t.

The tangent line to $\mathcal{C}$ at $\vec{r}(a)$ has direction vector $\vec{r}\,'(a) = \langle f'(a), g'(a), h'(a) \rangle$.

The arclength of the section of $\mathcal{C}$ defined by the interval $[a, b]$ on $\vec{r}$ is $\displaystyle\int_a^b \left|\,\vec{r}\,'(\tau)\,\right| d\tau$.

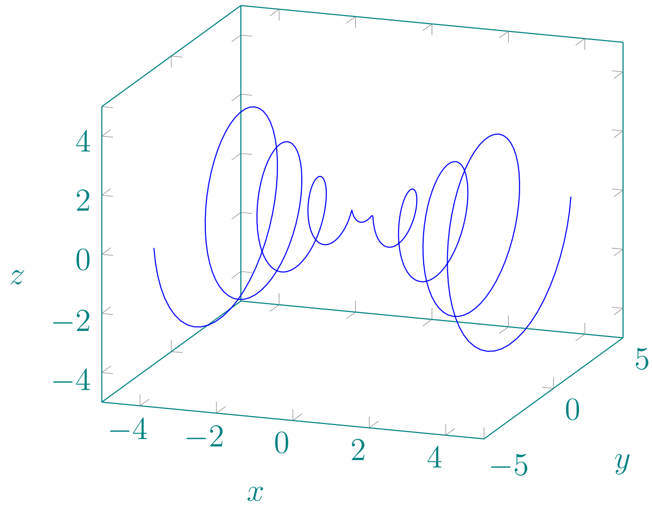
Space curves are visualized by projecting $\mathcal{C}$ onto planes and determining surfaces on which $\mathcal{C}$ lies, that is, finding equations satisfied by $\vec{r}$.

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \right\rangle \quad \text{with distinct antiderivative constants in components.}$$

1. Sketch the graph of each curve by finding surfaces on which they lie.

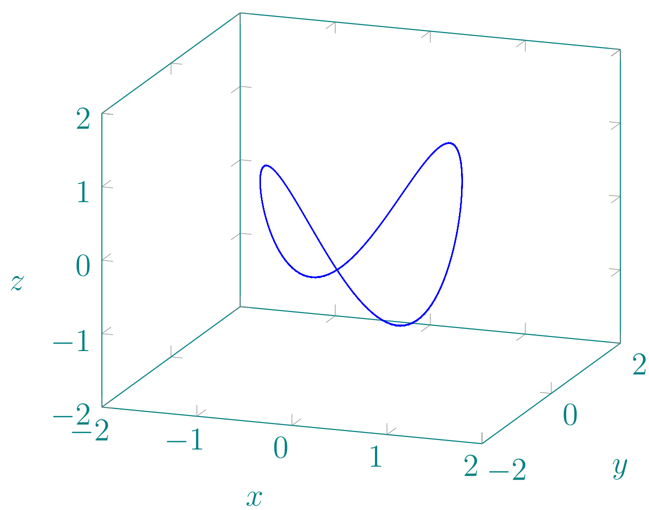
   (A) $\langle t, t\cos(t), t\sin(t) \rangle$ (B) $\langle \cos(t), \sin(t), \sin(2t) \rangle$ (C) $\langle t, \cos(t), \sin(t) \rangle$

   (A) $\langle t, t\cos(t), t\sin(t) \rangle$



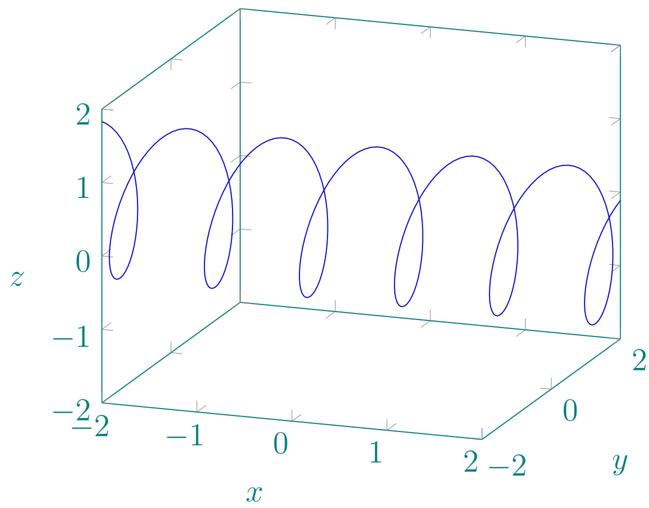
The curve lies on the cone $x^2 = y^2 + z^2$.

(B) $\langle \cos(t), \sin(t), \sin(2t) \rangle$



The curve lies on the cylinder $x^2 + y^2 = 1$. Through one rotation around the cylinder, the curve goes up and down twice.

(C) $\langle t, \cos(t), \sin(t)\rangle$



The curve lies on the cylinder $y^2 + z^2 = 1$.

2. Find a vector function that represents the intersection of the surfaces $x^2 + y^2 = 4$ and $z = xy$.

First, parametrize the cylinder $x^2 + y^2 = 4$ using $x(t) = 2\cos(t)$ and $y(t) = 2\sin(t)$. To satisfy the second equation, $z(t) = 4\cos(t)\sin(t)$.

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(t)\sin(t)\rangle$$

Note: An infinite number of distinct solutions exist.

3. Find the tangent line to the curve $\vec{r}(t) = \langle 2\cos(2\pi t), 2\sin(2\pi t), 4t\rangle$ at $(0, 2, 1)$.

The curve hits $(0, 2, 1)$ at $t = \dfrac{1}{4}$.

$$\vec{r}\,'(t) = \langle -4\pi\sin(2\pi t), 4\pi\cos(2\pi t), 4\rangle \qquad \vec{r}\,'(1/4) = \langle -4\pi, 0, 4\rangle$$

The tangent line to $\vec{r}$ at $t = 1/4$ is $\overrightarrow{L}(t) = \langle 0, 2, 1\rangle + t\,\langle -4\pi, 0, 4\rangle$.

Optimization: (Sections 14.7 and 14.8)

Optimization - Absolute Extrema: The *Extreme Value Theorem* guarantees that functions which are continuous on a *closed* and *bounded* set $\mathcal{D}$ attain an absolute maximum and minimum value in $\mathcal{D}$.

Absolute extrema of a continuous function on a closed and bounded set are located using the *Closed Interval Method*.

(I) Find all critical points in $\mathcal{D}$ and their values.

(II) Find the values of the absolute extrema of f on the boundary of $\mathcal{D}$ using either

(a) substitution and the closed interval method from MATH 125,

or (b) Lagrange Multipliers.

(III) The largest values from (I) and (II) are the absolute maximum values and the smallest values are the absolute minimum values.

Lagrange Multipliers: If f and g are differentiable functions and f has a local extrema on the constraint curve $g(x, y) = k$ at (a, b), where $\nabla g(a, b) \neq \vec{0}$, then there exists a scalar λ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

Exercises:

1. Find the minimum distance from the cone $z = \sqrt{x^2 + y^2}$ to the point $(-6, 4, 0)$.

Minimize the square root of the distance from (x, y, z) to $(-6, 4, 0)$

$$D(x, y, z) = (x + 6)^2 + (y - 4)^2 + z^2$$

Constraints: $g(x, y, z) = x^2 + y^2 - z^2 = 0$ and $z \geq 0$.

Using Lagrange Multipliers, $\nabla D = \lambda \nabla g$.

$$2x + 12 = \lambda 2x \qquad 2y - 8 = \lambda 2y \qquad 2z = -\lambda 2z \qquad z = \sqrt{x^2 + y^2}$$

From the third equation, either $\lambda = -1$ or $z = 0$.

If $z = 0$, then $(x, y, z) = (0, 0, 0)$. If $\lambda = -1$, then $(x, y, z) = (-3, 2, \sqrt{13})$.

Since $D(0, 0, 0) = 52$ and $D(-3, 2, \sqrt{13}) = 26$, the minimum distance is $\sqrt{26}$ at the point $(-3, 2, \sqrt{13})$.

2. A cardboard box without a lid is to have a volume of $32\, in^3$. Find the dimensions that minimizes the amount of cardboard used.

Minimize the surface area of the box with dimensions x, y, z.

$$S(x, y, z) = xy + 2xz + 2yz$$

Constraint: $V(x, y, z) = xyz = 32$.

Using Lagrange Multipliers, $\nabla S = \lambda \nabla V$.

$$y + 2z = \lambda yz \qquad x + 2z = \lambda xz \qquad 2x + 2y = \lambda xy \qquad xyz = 32$$

From the first three equations, $xy + 2xz = xy + 2yz = 2xz + 2yz = \lambda xyz$, implying $x = y = 2z$.

Therefore, $x(x)(0.5x) = 32$ and $(x, y, z) = (4, 4, 2)$.

3. Find the point closest to the origin on the line of intersection of the planes $y+2z = 12$ and $x+y = 6$.

Minimize the square root of the distance from (x, y, z) to $(0, 0, 0)$

$$D(x, y, z) = x^2 + y^2 + z^2$$

Constraints: $g(x, y, z) = y + 2z = 12$ and $h(x, y, z) = x + y = 6$.

Using Lagrange Multipliers, $\nabla D = \lambda \nabla g + \mu \nabla h$.

$$2x = \mu \qquad 2y = \lambda + \mu \qquad 2z = 2\lambda \qquad y + 2z = 12 \qquad x + y = 6$$

From the first three equations, $2y = z + 2x$.

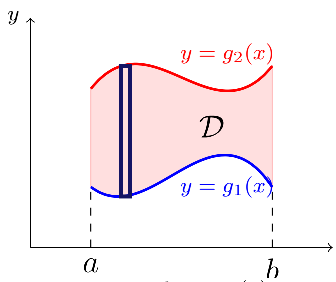
Solving the system of equations yields $(x, y, z) = (2, 4, 4)$.

Double and Triple Integrals: (Sections 15.1, 15.2, 15.3)

$\iint_{\mathcal{D}} f(x, y)\, dA$ represents the net volume contained under the surface $z = f(x, y)$ over the domain $\mathcal{D}$.

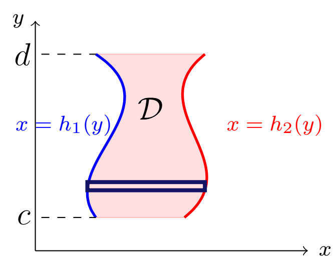
Fubini's Theorem: If $\mathcal{R} = [a, b]\times[c, d]$, then $\iint_{\mathcal{R}} f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$.

Vertically Simple Regions



$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Horizontally Simple Regions

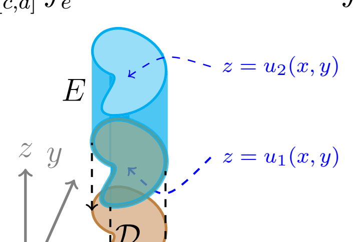


$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

$\iiint_{\mathcal{S}} f(x, y)\, dV$ represents the net hypervolume contained under $t = f(x, y, z)$ over the solid $\mathcal{S}$.

Fubini's Theorem: If $\mathcal{S} = [a, b] \times [c, d] \times [e, f]$, then $\iiint_{\mathcal{S}} f(x, y, z)\, dV =$

$$\iint_{[a,b]\times[c,d]} \int_e^f f(x, y, z)\, dz\, dA = \iint_{[a,b]\times[e,f]} \int_c^d f(x, y, z)\, dy\, dA = \iint_{[c,d]\times[e,f]} \int_a^b f(x, y, z)\, dx\, dA$$

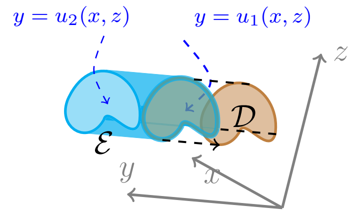


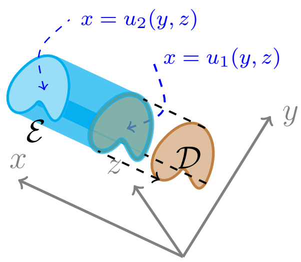
Z-Simple Solids:

$$\iiint_{\mathcal{E}} f(x,y,z)\,dV \quad = \iint_{\mathcal{D}} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$$

Y-Simple Solids:

$$\iiint_{\mathcal{E}} f(x,y,z)\,dV \quad = \iint_{\mathcal{D}} \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\,dA$$





X-Simple Solids:

$$\iiint_{\mathcal{E}} f(x,y,z)\,dV \quad = \iint_{\mathcal{D}} \int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dA$$

The moments and the center of mass:

The coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\delta(x,y)\,dA$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\delta(x,y)\,dA$$

The coordinates $(\overline{x}, \overline{y}, \overline{z})$ of the center of mass of a solid are

$$\overline{x} = \frac{M_{yz}}{m} = \frac{1}{m}\iiint_S x\delta(x,y,z)\,dV$$

$$\overline{y} = \frac{M_{xz}}{m} = \frac{1}{m}\iiint_S y\delta(x,y,z)\,dV$$

$$\overline{z} = \frac{M_{xy}}{m} = \frac{1}{m}\iiint_S z\delta(x,y,z)\,dV$$

Exercises:

1. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

The solid consists of points (x,y,z) where $0 \le z \le 4 - x^2 - y$, $0 \le y \le 4 - x^2$, $0 \le x \le 2$.

$$\int_0^2 \int_0^{4-x^2} \int_0^{4-x^2-y} 1\,dz\,dy\,dx = \int_0^2 \int_0^{4-x^2} 4 - x^2 - y\,dy\,dx = \frac{1}{2}\int_0^2 (4-x^2)^2\,dx = \frac{128}{15}$$

2. For the following iterated integrals, sketch the region of integration, write an equivalent double integral, and evaluate the integral.

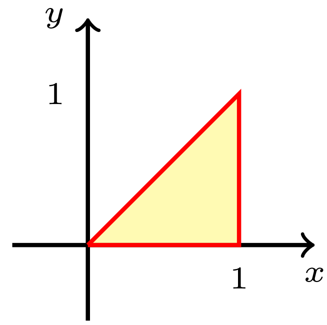
(A) $\int_0^1 \int_y^1 x^2 e^{xy}\,dx\,dy$

(B) $\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\,dy\,dx$

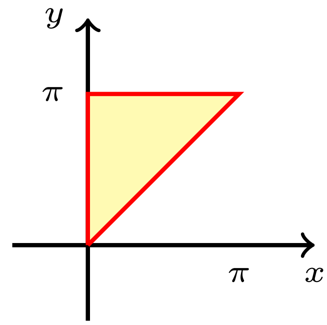
(C) $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$

(D) $\int_0^2 \int_x^2 2y^2\sin(xy)\,dy\,dx$

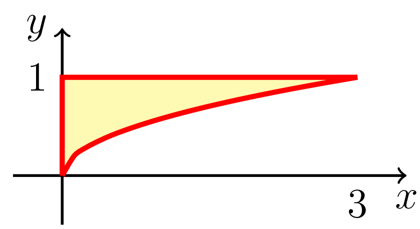
(A) $$\int_0^1 \int_y^1 x^2 e^{xy}\,dx\,dy = \int_0^1 \int_0^x x^2 e^{xy}\,dy\,dx = \int_0^1 xe^{x^2} - x\,dx = \frac{1}{2}(e-2)$$



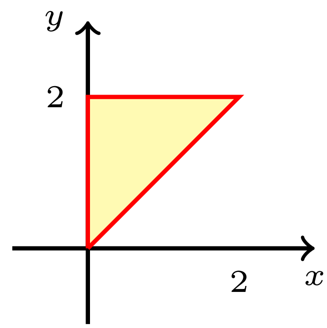
(B) $$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}\,dy\,dx = \int_0^\pi \int_0^y \frac{\sin(y)}{y}\,dx\,dy = \int_0^\pi \sin(y)\,dy = 2$$



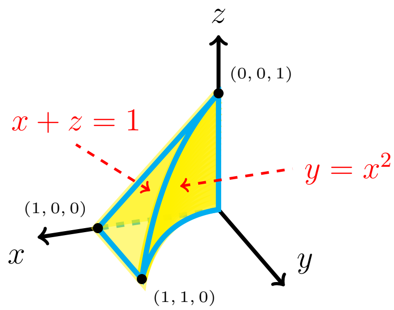
(C) $$\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx = \int_0^1 \int_0^{3y^2} e^{y^3}\,dx\,dy = \int_0^1 3y^2 e^{y^3}\,dy = e-1$$



(D) $$\int_0^2 \int_x^2 2y^2\sin(xy)\,dy\,dx = \int_0^2 \int_0^y 2y^2\sin(xy)\,dx\,dy = \int_0^2 2y - 2y\cos(y^2)\,dy = 4 - \sin(4)$$

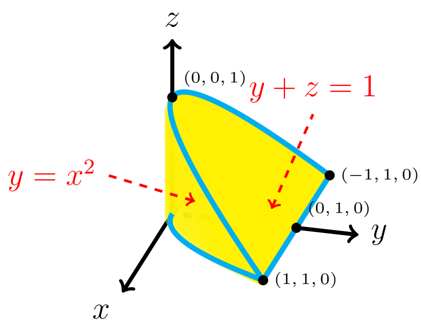


3. Let $\mathcal{S}$ be the solid in the *first octant* defined by $z = 1 - x$ and $y = x^2$. Iterate $\iiint_{\mathcal{S}} f(x,y,z)\,dV$ in three ways: dz first, dy first, and dx first.



$$\int_0^1 \int_0^{x^2} \int_0^{1-x} f\,dz\,dy\,dx \qquad \int_0^1 \int_0^{1-x} \int_0^{x^2} f\,dy\,dz\,dx \qquad \int_0^1 \int_0^{1-\sqrt{y}} \int_{\sqrt{y}}^{1-z} f\,dx\,dz\,dy$$

4. Let $\mathcal{S}$ be the solid defined by $y + z = 1$, $y = x^2$, and $z = 0$. Iterate $\iiint_{\mathcal{S}} f(x,y,z)\,dV$ in three ways: dz first, dy first, and dx first.



$$\int_{-1}^1 \int_{x^2}^{1} \int_0^{1-y} f\,dz\,dy\,dx \qquad \int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} f\,dy\,dz\,dx \qquad \int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} f\,dx\,dz\,dy$$

Change of Variables: (Sections 12.7, 15.4, 15.6)

A transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is an invertible map $T(u,v) = (x,y)$ where $x = f(u,v)$ and $y = g(u,v)$ have continuous first-order partial derivatives.

The Jacobian of the transformation $T(u,v) = \big(x(u,v), y(u,v)\big)$ represents the instantaneous change in area near (u,v).

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

If $T(\mathcal{S}) = \mathcal{R}$, then $\displaystyle\iint_{\mathcal{R}} f(x,y)\,dA_{xy} = \iint_{\mathcal{S}} f\big(x(u,v), y(u,v)\big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$.

Change of Variables should be used to simplify the integrand or simplify the domain of integration.

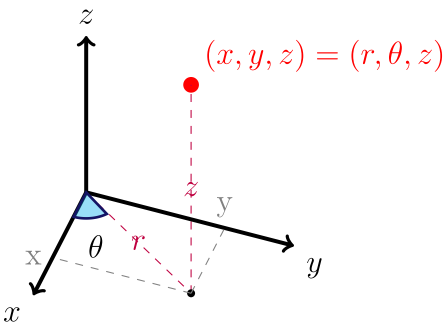
Common Transformations:

Polar Coordinates: $G(r,\theta) = \big(r\cos(\theta), r\sin(\theta)\big)$ with $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$.

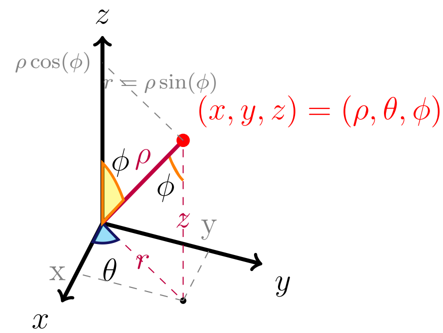
Cylindrical Coordinates: $G(r,\theta,z) = \big(r\cos(\theta), r\sin(\theta), z\big)$ with $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = r$.

Spherical Coordinates: $G(\rho,\phi,\theta) = \big(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\big)$ with $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin(\phi)$.

Cylindrical Coordinates



Spherical Coordinates



Exercises:

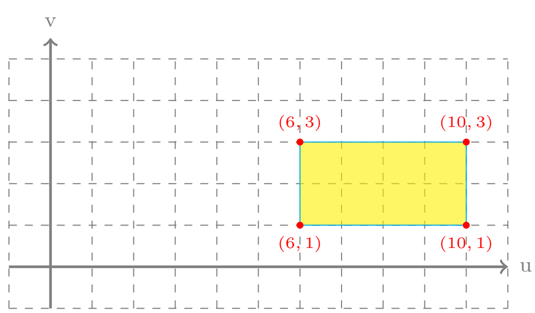
1. Evaluate $\displaystyle\iint_{\mathcal{R}} x + 3y\,dA$ where $\mathcal{R}$ is defined by $x+2y=10$, $x+2y=6$, $y=1$, and $y=3$.

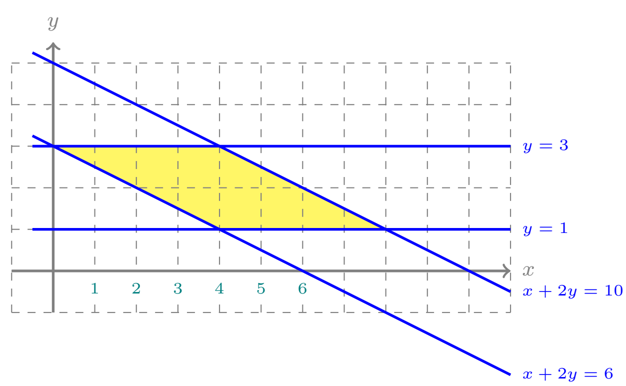
Using the transformation $T^{-1}(x,y) = (x+2y, y)$, the region $\mathcal{R}$ is mapped to the region consisting of points (u,v) where $6 \le u \le 10$ and $1 \le v \le 3$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \qquad \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Since $x + 3y = x + 2y + y = u + v$,

$$\iint_{\mathcal{R}} x + 3y\,dA = \int_1^3 \int_6^{10} (u+v)\,1\,du\,dv = 80$$





2. Evaluate $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}}\, dx\, dy$.

The region of integration consists of the points (x, y) where $1 \leq y \leq 2$ and $1/y \leq x \leq y$.

This region is bounded by the curves $y = 2$, $y = x$, and $y = \dfrac{1}{x}$.

Using the transformation $T^{-1}(x, y) = \left(\dfrac{y}{x}, xy\right)$, the boundaries of the region are transformed to the curves $v = \dfrac{4}{u}$, $v = 1$, and $u = 1$ in the uv-plane.

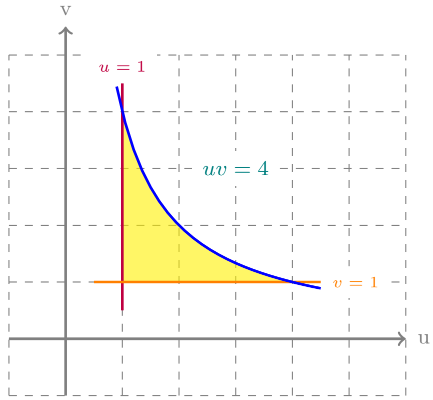
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ y & x \end{vmatrix} = \frac{-2y}{x} = -2u \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{-2u}$$

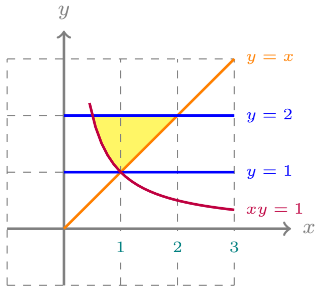
$$\begin{aligned} \int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}}\, dx\, dy &= \int_1^4 \int_1^{4/v} \sqrt{u} e^{\sqrt{v}} \frac{1}{2u}\, du\, dv \\ &= \int_1^4 \left(\sqrt{\frac{4}{v}} - 1\right) e^{\sqrt{v}}\, dv \approx 8.08 \end{aligned}$$

Note:

Use $U = 2 - \sqrt{v}$ and $dV = \dfrac{e^{\sqrt{v}}}{\sqrt{v}} dv$ for integration parts to get:

$$\int \left(\sqrt{\frac{4}{v}} - 1\right) e^{\sqrt{v}}\, dv = -2e^{\sqrt{(v)}}(-3 + \sqrt{(v)}) + c$$





3. Evaluate $\displaystyle\iint_{\mathcal{R}} (x - y)^2 \sin^2(x + y)\, dA$ where $\mathcal{R} = [\pi, 2\pi] \times [0, \pi]$.

Use the transformation $T^{-1}(x, y) = (x - y, x + y)$, where $T(u, v) = \left(\dfrac{u + v}{2}, \dfrac{v - u}{2}\right)$.

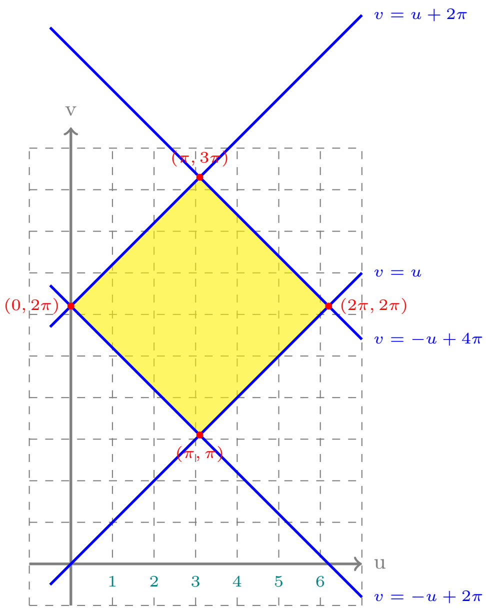
- The edge $x = \pi$ is mapped to $v = -u + 2\pi$.

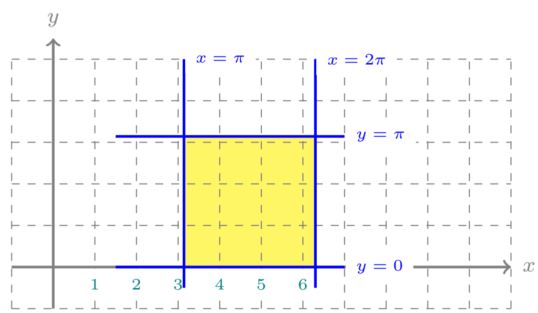
- The edge $x = 2\pi$ is mapped to $v = -u + 4\pi$.
- The edge $y = 0$ is mapped to $v = u$.
- The edge $y = \pi$ is mapped to $v = u + 2\pi$.

The region $\mathcal{R}$ is mapped to the region consisting of points (u, v) where $-u + 2\pi \leq v \leq u + 2\pi$ where $0 \leq u \leq \pi$ or $u \leq v \leq -u + 4\pi$ where $\pi \leq u \leq 2\pi$.

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{vmatrix} = \frac{1}{2}$$

$$\begin{aligned}\iint_{\mathcal{R}} (x - y)^2 \sin(x + y)^2 \, dA &= \frac{1}{2} \int_0^{\pi} \int_{-u+2\pi}^{u+2\pi} u^2 \sin^2(v) \, dv \, du + \frac{1}{2} \int_{\pi}^{2\pi} \int_{u}^{-u+4\pi} u^2 \sin^2(v) \, dv \, du \\ &= \frac{1}{8} \left(\pi^4 + \pi^2\right) + \frac{1}{24} \left(11\pi^4 - 9\pi^2\right)\end{aligned}$$





(x, y)	(u, v)
$(\pi, 0)$	(π, π)
(π, π)	$(0, 2\pi)$
$(2\pi, \pi)$	$(\pi, 3\pi)$
$(2\pi, 0)$	$(2\pi, 2\pi)$

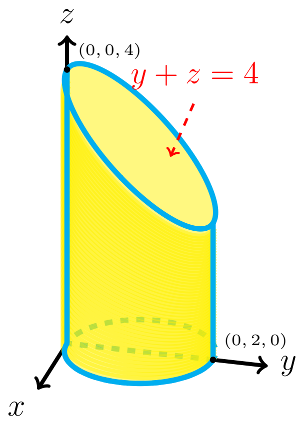
4. Evaluate $\iint_{\mathcal{R}} 3x^2 + 2x \, dA$ where $\mathcal{R}$ is the region bounded by the curves $y = x^3 + 6$, $y = x^3 + 5$, $y = 9 - x^2$, and $y = 8 - x^2$.

Using the transformation $T^{-1}(x, y) = (y - x^3, y + x^2)$, the region $\mathcal{R}$ is mapped to the rectangle in the uv-plane consisting of points (u, v) where $5 \leq u \leq 6$ and $8 \leq v \leq 9$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} -3x^2 & 1 \\ 2x & 1 \end{vmatrix} = -3x^2 - 2x$$

$$\iint_{\mathcal{R}} 3x^2 + 2x \, dA = \int_5^6 \int_8^9 3x^2 + 2x \left| \frac{1}{-3x^2 - 2x} \right| dv \, du = 1$$

5. Find the volume of the solid bounded by $z = 0$, $z = 4 - y$, and $x^2 + (y-1)^2 = 1$.



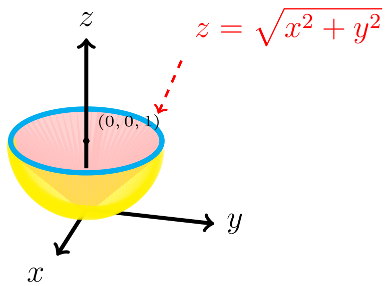
Using polar coordinates, the equation $x^2 + (y-1)^2 = 1$ is $r = 2\sin(\theta)$.

The volume of the solid is

$$\iint_{x^2+(y-1)^2=1} 4 - y \, dA = \int_0^{\pi} \int_0^{2\sin(\theta)} (4 - r\sin(\theta))\, r \, dr \, d\theta = 3\pi$$

You can use cylindrical coordinates a triple integral to get the same result.

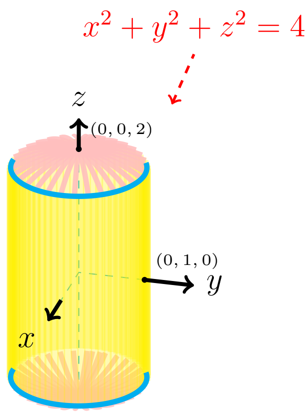
6. Find the volume of the solid above the sphere $x^2 + y^2 + z^2 = 2z$ and below the cone $z = \sqrt{x^2 + y^2}$.



Using spherical coordinates, the volume of the solid is

$$\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos(\phi)} 1 \, \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{\pi}{3}$$

7. Find the volume of the region cut from the cylinder $x^2 + y^2 = 1$ by the sphere $x^2 + y^2 + z^2 = 4$.



Using cylindrical coordinates, the volume of the solid is

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1\, r\, dz\, dr\, d\theta = \frac{4\pi}{3}\left(8-3\sqrt{3}\right)$$

Vector Fields:

(Sections 16.1)

A vector field in $\mathbb{R}^n$, denoted $\vec{F}$, is a function that assigns to each point $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ a vector $\vec{F}(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$. The vector field field $\vec{F}$ is smooth if each of its' components are continuously differentiable.

A vector field $\vec{F}$ is a unit vector field if $\left\|\vec{F}(P)\right\| = 1$ for every point P.

A vector field $\vec{F}$ is a radial vector field if $F(P)$ depends only on the distance from P to the origin, O, and is parallel to $\vec{OP}$.

$$\vec{E}_{\mathbb{R}^2} = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle \qquad \vec{E}_{\mathbb{R}^3} = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

The divergence of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is defined

$$\text{div}\left(\vec{F}\right) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

The curl of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is defined

$$\text{curl}\left(\vec{F}\right) = \nabla \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Given a differential function $f(x, y, z)$, its gradient vector field

$$\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

is called a conservative vector field. The function f is called a potential function for $\vec{F}$.

If the vector field $\vec{F} = \langle F_1, F_2 \rangle$ is conservative then $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$.

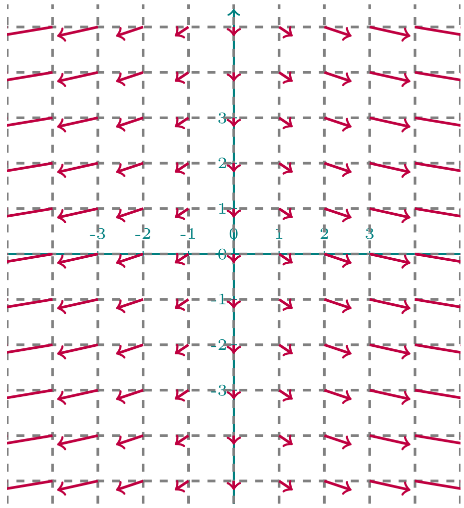
If the vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is conservative then curl $\left(\vec{F}\right) = \vec{0}$ and

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

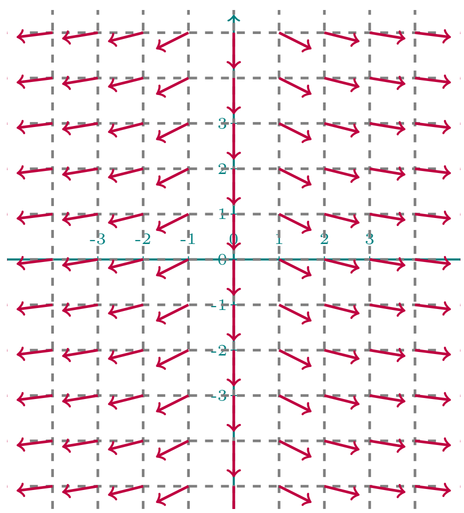
Exercises:

1. $f(x, y) = x^2 - y$ is a potential function for $\vec{F}$. Find and sketch $\vec{F}$.

$\vec{F}(x, y) = \langle 2x, -1 \rangle$

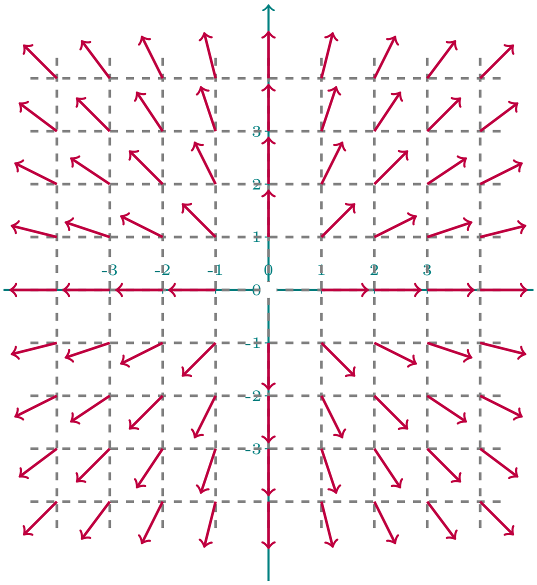


OR



2. $f(x, y) = \sqrt{x^2 + y^2}$ is a potential function for $\vec{F}$. Find and sketch $\vec{F}$.

$$\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$



3. Calculate the curl and divergence of the vector fields:

   (A) $\vec{F}(x,y,z) = \langle xyz, 0, -x^2y \rangle$

$$\operatorname{div}\left(\vec{F}\right) = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-x^2y) = yz$$

$$\operatorname{curl}\left(\vec{F}\right) = \left\langle \frac{\partial}{\partial y}(-x^2y) - \frac{\partial}{\partial z}(0), \frac{\partial}{\partial z}(xyz) - \frac{\partial}{\partial x}(-x^2y), \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(xyz) \right\rangle = \langle -x^2, 3xy, -xz \rangle$$

   (B) $\vec{F}(x,y,z) = \langle 0, \cos(xz), -\sin(xy) \rangle$

$$\operatorname{div}\left(\vec{F}\right) = 0 \qquad \operatorname{curl}\left(\vec{F}\right) = \langle -x\cos(xy) + x\sin(xz), y\cos(xy), -z\sin(xz) \rangle$$

   (C) $\nabla\left(e^{xyz}\right)$

$$\nabla\left(e^{xyz}\right) = \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle$$

$\vec{F}$ is conservative, so $\operatorname{curl}\left(\vec{F}\right) = \vec{0}$

$$\operatorname{div}\left(\vec{F}\right) = \left(y^2z^2 + x^2z^2 + x^2y^2\right)e^{xyz}$$